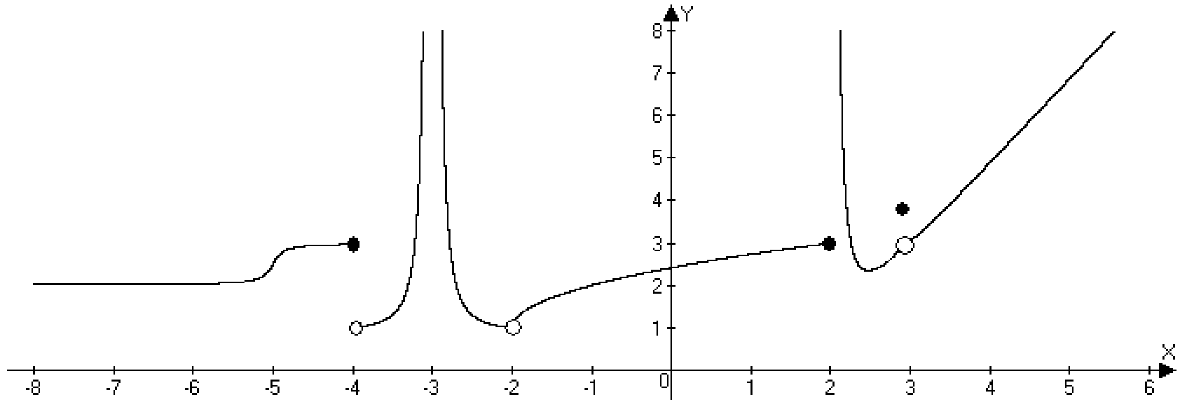
Math 1A – Chapter 2 Test Review Problems Solutions.

1. The graph of f is given.



a. Find each limit, or explain why it doesn't exist:

i. $\lim_{x\to -4^-} f(x) = 3$ ii. $\lim_{x\to -4^+} f(x) = 1$ iii. $\lim_{x\to -4} f(x)$ doesn't exist.

iv. $\lim_{x\to -2} f(x) = 1$ v. $\lim_{x\to 2^+} f(x) = \infty$ vi. $\lim_{x\to 3} f(x) = 3$

b. Where does the function have removable discontinuities?
ANS: There are removable discontinuities where $x = -2$ and $x = 3$.

c. Where does the function have a jump discontinuity?
ANS: The function has a jump discontinuity where $x = -4$.

2. Find the limit

a. $\lim_{x\to 3}\frac{x^3-27}{x^4-81} = \lim_{x\to 3}\frac{x^2+3x+9}{x^3+3x^2+9x+27} = \frac{1}{4}$

b. $\lim_{x\to\infty}\frac{x-2}{\sqrt{3x^2-x}}\frac{\div x}{\div\sqrt{x^2}} = \lim_{x\to\infty}\frac{1-\frac{2}{x}}{\sqrt{3-\frac{1}{x}}} = \frac{\sqrt{3}}{3}$

c. $\lim_{x\to\infty}\frac{1}{1-e^{-x}} = \frac{1}{1-\frac{1}{\lim_{x\to\infty} e^x}} = \frac{1}{1-0} = 1$

d. $\lim_{x\to 1^-}\frac{1}{\ln|x-1|} = \frac{1}{\ln\left|\lim_{x\to 1^-} x-1\right|} = \frac{1}{\ln 0^+} = \frac{1}{-\infty} = 0$

3. Use the intermediate value theorem to prove that $x^2 = 2^x$ has a solution in (-1,0).
SOLN: Consider $f(x) = 2^x - x^2$. Since f is a sum of continuous functions, it is continuous.
Also 0 is between $f(-1) = -\frac{1}{2}$ and $f(0) = 1$ so by the Intermediate Value Theorem, there is $c \in (-1,0)$ such that $f(c) = 0$, which is a solution to given equation.

4. If the tangent to $y = f(x)$ at (5,4) passes through the point (1,2), find $f(5)$ and $f'(5)$.

SOLN: $f(5) = 4$ is given. The slope of the tangent line is $f'(5) = \frac{4-2}{5-1} = \frac{1}{2}$

5. Find the derivative of $f(x)=\frac{3+x}{1-3x}$ using the definition of the derivative.

SOLN: One way is to plug directly into the formula for the limit of a difference quotient:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{3+x+h}{1-3x-3h}-\frac{3+x}{1-3x}}{h}=$$

$$=\lim_{h\to 0}\frac{(3+x+h)(1-3x)-(3+x)(1-3x-3h)}{h(1-3x-3h)(1-3x)}=$$

$$=\lim_{h\to 0}\frac{(3+x)(1-3x)+h(1-3x)-(3+x)(1-3x)+3h(3+x)}{h(1-3x-3h)(1-3x)}=\lim_{h\to 0}\frac{h(1-3x)+3h(3+x)}{h(1-3x-3h)(1-3x)}$$

$$=\lim_{h\to 0}\frac{(1-3x)+3(3+x)}{(1-3x-3h)(1-3x)}=\frac{10}{(1-3x)^2}$$

Alternatively, it's a bit simpler to do division to write $f(x)=\frac{3+x}{1-3x}=-\frac{1}{3}+\frac{10/3}{1-3x}$ and then plug into the difference quotient:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{-\frac{1}{3}+\frac{10}{3(1-3x-3h)}+\frac{1}{3}-\frac{10}{3(1-3x)}}{h}=$$

$$=\lim_{h\to 0}\frac{10(1-3x)-10(1-3x-3h)}{3h(1-3x)(1-3x-3h)}=\lim_{h\to 0}\frac{-30x+30x+30h}{3h(1-3x)(1-3x-3h)}=\frac{10}{(1-3x)^2}$$

6. Suppose that we don't have a formula for $g(x)$ but we know that $g(2) = -4$ and $g'(x)=\sqrt{x^2+5}$ for all x.

   a. Use a linear approximation to estimate $g(1.95)$ and $g(2.05)$.

   SOLN: Using the line tangent to $g(x)$ at $x = 2$, that is, $g(x)\approx g(2)+g'(2)(x-2)=-4+3(x-2)$ we have $g(1.95)\approx -4+3(-0.05)=-4.15$ and $g(2.05)\approx -4+3(0.05)=-3.85$

   b. Are your estimates in part (a) too large or too small? Explain.

   SOLN: Since $g'(x)=\sqrt{x^2+5}$ is increasing, $g(x)$ is concave up and so the tangent line lies beneath the curve. Therefore both estimates are underestimates.

7. Is there a number a such that $\lim_{x\to -2}\frac{2x^2+ax+a}{x^2+x-2}$ exists? If not, why not? If so, find the value of a and the value of the limit.

SOLN: $\lim_{x\to -2}\frac{2x^2+ax+a}{x^2+x-2}=\lim_{x\to -2}\frac{2x^2+ax+a}{(x+2)(x-1)}$ can exist only if the numerator goes to zero at -2.

That is $2(-2)^2+a(-2)+a=0\Leftrightarrow a=8$. If $a = 8$ then

$$\lim_{x\to -2}\frac{2x^2+8x+8}{x^2+x-2}=\lim_{x\to -2}\frac{2(x+2)^2}{(x+2)(x-1)}=\lim_{x\to -2}\frac{2(x+2)}{x-1}=0$$

8. Consider $\lim_{x\to 0}\ln\left(x+\cos\left(x\right)\right)$.

a. What theorem is essential to evaluating this limit. Why are the conditions of the theorem met?
SOLN: The relevant theorem says that if $\lim_{x\to a} g\left(x\right)=b$ (i.e. the limit exists) and if *f* is continuous at *b* (i.e. $\lim_{x\to b} f\left(x\right)=f\left(b\right)$) then $\lim_{x\to a} f\left(g\left(x\right)\right)=f\left(\lim_{x\to a} g\left(x\right)\right)=f\left(b\right)$. Since $f\left(x\right)=\ln\left(x+\cos\left(x\right)\right)$ is a composition of continuous functions, it is also continuous, thus the conditions of theorem are met.

b. Use the theorem to evaluate the limit.

SOLN: $$\lim_{x\to\pi/2}\ln\left(x+\cos\left(x\right)\right)=\ln\left(\lim_{x\to 0}\left(x+\cos\left(x\right)\right)\right)=\ln\left(\lim_{x\to 0}x+\lim_{x\to 0}\cos\left(x\right)\right)=$$
$$=\ln\left(0+\cos\left(\lim_{x\to 0}x\right)\right)=\ln\left(1\right)=0$$

9. For the function $f\left(x\right)$ whose derivative function $f'\left(x\right)$ is graphed below,

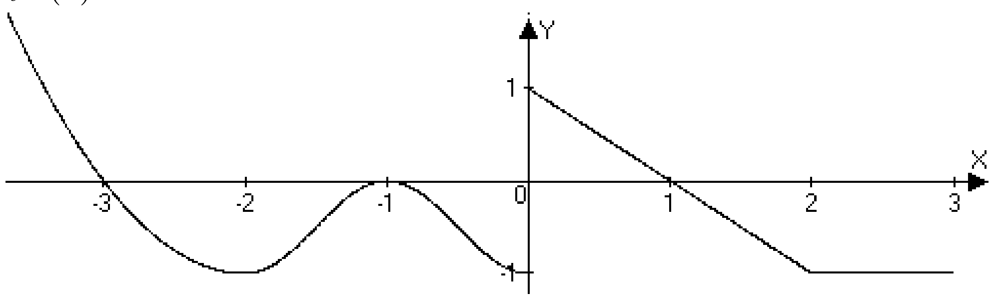
a. $f\left(x\right)$ is increasing on $\left(-4,-3\right)\cup\left(0,1\right)$

b. $f\left(x\right)$ is concave up on (–2,1)

c. $f\left(x\right)$ has a local maximum where x = -3 and where x = 1.

d. $f''\left(x\right)$ is positive on (–2,1)

e. $f''\left(x\right)=0$ where $x=-2$ and where $x=-1$.



10. Consider $f\left(x\right)=\dfrac{\sqrt{x}-8}{\sqrt[3]{x}-4}$

a. Approximate the value of $f\left(x\right)$ at x = 64.001 and 63.999 – what do your results suggest about $\lim_{x\to 64} f\left(x\right)$?

SOLN:

$$f\left(64.001\right)=\frac{\sqrt{64.001}-8}{\sqrt[3]{64.001}-4}\approx\frac{0.00006249976}{0.00002093322}\approx 3.000003907$$

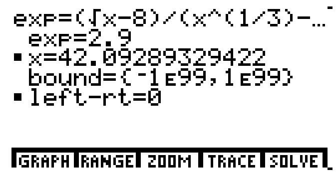
$$f\left(63.999\right)=\frac{\sqrt{63.999}-8}{\sqrt[3]{63.999}-4}\approx\frac{-0.000062500244}{-0.000020833442}\approx 3.0000039072$$

Which strongly suggests that $\lim_{x\to 64} f\left(x\right)=3$.

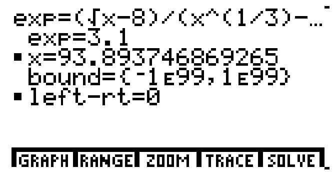
b. How close does x have to be to 64 to ensure that the function is within 0.1 of it's limit?
SOLN:

We want to choose a distance δ small enough to be sure that if

$$64-\delta < x < 64+\delta \text{ then } 2.9 < f(x) < 3.1 \Leftrightarrow 2.9 < \frac{\sqrt{x}-8}{\sqrt[3]{x}-4} < 3.1.$$



It turns out this gives great latitude for δ. It is appropriate to use the SOLVER on the TI85 (see screen shots below), by which we find that $f(x) = 2.9$ near $x = 42.0929 = 64 - 21.9071$ and $f(x) = 3.1$ near $x = 93.8937 = 64 + 29.8937$, so that, if x is not much farther than 21.9 units from 64, then $f(x)$ will be within 0.1 units of 3.



Note also that

$$f(x)=\frac{\sqrt{x}-8}{\sqrt[3]{x}-4}\frac{\sqrt[3]{x^2}+4\sqrt[3]{x}+16}{\sqrt[3]{x^2}+4\sqrt[3]{x}+16}=\frac{\left(\sqrt{x}-8\right)\left(\sqrt[3]{x^2}+4\sqrt[3]{x}+16\right)}{x-64}\frac{\sqrt{x}+8}{\sqrt{x}+8}=\frac{\sqrt[3]{x^2}+4\sqrt[3]{x}+16}{\sqrt{x}+8}$$

11. Is there a number a such that $\lim_{x\to 1}\frac{2x^2+ax+a}{x^2+x-2}$ exists? If not, why not? If so, find the value of a and the value of the limit.

SOLN: Since the denominator of $\frac{2x^2+ax+a}{x^2+x-2}$ has a zero at $x = 1$, we need to require that the numerator also has a zero at $x = 1$; that is, $2+a+a = 0$ or $a = -1$. To be sure,

$$\lim_{x\to 1}\frac{2x^2-x-1}{x^2+x-2}=\lim_{x\to 1}\frac{(2x+1)(x-1)}{(x+2)(x-1)}=\lim_{x\to 1}\frac{2x+1}{x+2}=1$$

12. Consider $\lim_{x\to\pi/2}\cos\left(2x+\cos(3x)\right)$.

a. What theorem is essential to evaluating this limit. Why are the conditions of the theorem met?

The relevant theorem says that if $\lim_{x\to a} g(x)=b$ (i.e. the limit exists) and if f is continuous at b (i.e. $\lim_{x\to b} f(x)=f(b)$) then $\lim_{x\to a} f\left(g(x)\right)=f\left(\lim_{x\to a} g(x)\right)=f(b)$. Since $f(x)=\cos\left(2x+\cos(3x)\right)$ is a composition of continuous functions, it is also continuous, thus the conditions of theorem are met.

b. Use the theorem to evaluate the limit.

SOLN:
$$\lim_{x\to\pi/2}\cos\left(2x+\cos(3x)\right)=\cos\left(\lim_{x\to\pi/2}\left(2x+\cos(3x)\right)\right)=\cos\left(\lim_{x\to\pi/2}2x+\lim_{x\to\pi/2}\cos(3x)\right)=$$
$$=\cos\left(\pi+\cos\left(\lim_{x\to\pi/2}3x\right)\right)=\cos(\pi+0)=-1$$

13. For the function g whose graph is shown, approximate the following, writing "DNE" if the limit doesn't exist and ∞ or $-\infty$, as appropriate.

a. $\lim_{x\to-\infty} g(x)=1$ horizontal asymptote

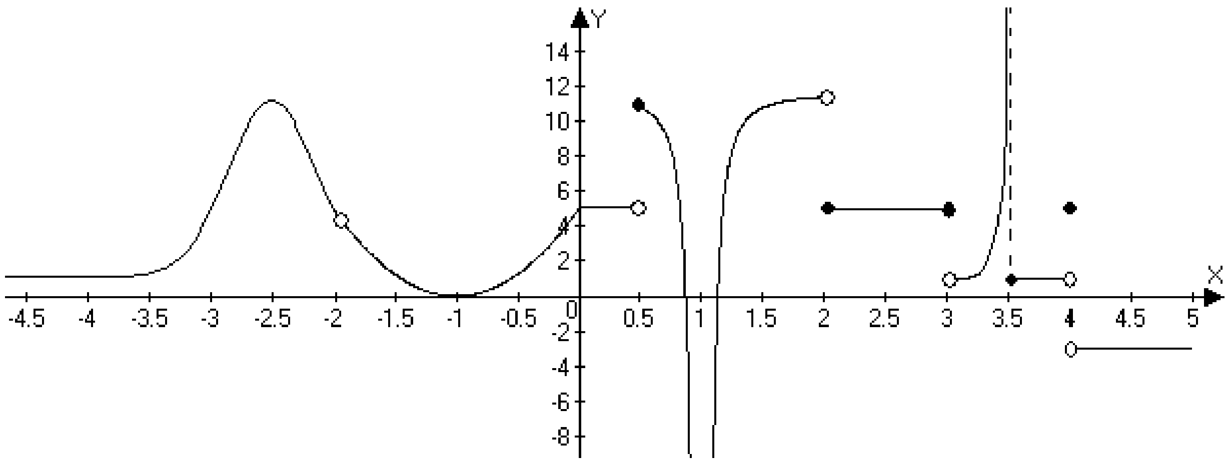
b. $\lim_{x\to-2} g(x)=4$

c. $\lim_{x\to 0.5^+} g(x)=11$

d. $\lim_{x\to 1} g(x)=-\infty$, DNE

e. $\lim_{x\to 2^-} g(x)=11$

f. $\lim_{x\to 3.5^+} g(x)=1$



14. Suppose the height H of an object (in meters) at time t (in seconds) is given by

$$H(t)=\begin{cases}0 & \text{if} \quad t<0 \\ 1 & \text{if} \quad t\geq 0\end{cases}$$

a. What is the average velocity over the interval $-1\leq t\leq 1$

SOLN: $\dfrac{H(1)-H(-1)}{1-(-1)}=\dfrac{1-0}{2}=\dfrac{1}{2}$ m/s.

b. Find an interval over which the average velocity of the object is a 1000 m/s.
SOLN: Since the jump discontinuity has an instantaneous change which is infinite, it shouldn't be to hard to find a relatively puny rate of change like 1000 m/s. Fix one point at (0,1) and a variable point (t,0) preceding that point. Then the average rate of change is $\dfrac{1-0}{0-t}=-\dfrac{1}{t}=1000\Leftrightarrow \boxed{t=-0.001\text{sec}}$.

15. Let $B(t)$ be the number of Elbonian buffalo per capita at time t. The table below gives values of $B(t)$ as of June 30 of the specified year. What is your best approximation to the value of $B'(2000)$?

t	1998	1999	2000	2001	2002
$B(t)$	12.5	10.2	9.80	9.20	7.95

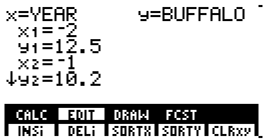
SOLN: Averaging the immediate before and after rates of change leads to

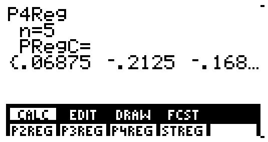
$$B'(2000)\approx\frac{\dfrac{B(2001)-B(2000)}{2001-2000}+\dfrac{B(200)-B(1999)}{2000-1999}}{2}=\frac{9.2-10.2}{2001-1999}=-\frac{1}{2}$$ buffalo per capita per year, but this doesn't take into account the data from 1998 and 2002.

Let's see if we can include these data. One way is to fit a quartic function to the data, which can be done by substituting the t, B pairs into the general form $B(t)\approx a_4t^4+a_3t^3+a_2t^2+a_1t+a_0$.

You could go ahead and solve the 4x4 system this leads to, or you could get the same result by using the TI85, say, to do "regression analysis" on the data. Start by entering the data using the STAT interface. See the first screen shot. Note that the t values are taken as years since 2000, ranging from -2 to 2. This is done to make the computations more stable. Then choose the P4REG calculation which shows that

$B(t-2000)\approx 0.06875t^4-0.2125t^3-0.16875t^2-0.2875t+9.8$ and thus in year 2000 we estimate $B'(2000) \approx -0.2875$ buffalo per capita per year, which is about half the rate of change computed with the previous method…but a whole lot more trouble! It can be argued that the extra trouble does produce a more accurate measure, but is there an easier method of computing it?





16. Consider the function $x(t) = \dfrac{1}{1+t^2}$.

a. Use the definition of the derivative to show that $x'(t) = \dfrac{-2t}{\left(1+t^2\right)^2}$. SOLUTION:

$$x'(t) = \lim_{h\to 0} \frac{\dfrac{1}{1+(t+h)^2} - \dfrac{1}{1+t^2}}{h} = \lim_{h\to 0} \frac{1+t^2-\left(1+(t+h)^2\right)}{h\left(1+(t+h)^2\right)\left(1+t^2\right)} = \lim_{h\to 0} \frac{t^2-t^2-2ht-h^2}{h\left(1+(t+h)^2\right)\left(1+t^2\right)} =$$

$$= \lim_{h\to 0} \frac{-2t-h}{\left(1+(t+h)^2\right)\left(1+t^2\right)} = \frac{-2t}{\left(1+t^2\right)^2}$$

b. Find an equation for the line tangent to $x(t)$ where $t = 1$.

SOLN: $y = \frac{1}{2} - \frac{1}{2}(x-1)$

c. Use a linear approximation to approximate $x(1.05)$

SOLN: $x(1.05) \approx 0.5 - 0.5(0.05) = 0.475$ Note that the true value is

$$x(1.05) = \frac{1}{1+1.1025} \approx 0.4756$$

17. For the function $f(x)$ whose derivative function $f'(x)$ is graphed below, find where:

a. $f(x)$ is increasing on (0,2)

b. $f(x)$ is concave up (-2,2) and (3,5)

c. $f(x)$ has a local maximum where $x = 2$.

d. $f''(x)$ is positive if f is the derivative is increasing: (-2,2) and (3,5)

e. $f''(x) = 0$ on the interval (-5,-2) and where $x = 3$ and $x = 5$.